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ARTICULATED ARM AND SPECIAL MULTI-FLAGS

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Abstract

In this paper, we give a kinematical illustration of some distributions called special multi-flags distributions. Precisely, we define the kinematic model in angular coordinates of an articulated arm constituted of a series of $(n+1)$ segments in $\mathbb{R}^{k+1}$ and construct the special multi-flag distribution associated to this model.

1. Introduction

The kinematic evolution of a car towing n trailers can be described by a Goursat distribution on the configuration space $M = \mathbb{R}^2 \times (\mathbb{S}^1)^{n+1}$. A Goursat distribution is a rank-$(l - s)$ distribution on a manifold M of dimension $l \geq 2 + s$, such that each element of its flag of Lie squares,

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$$D = D^s \subset D^{s-1} = [D^s, D^s] \subset \cdots \subset D^{j-1} = [D^j, D^j] \subset \cdots \subset D^0 = TM$$

is of codimension 1 in the following one.

Since 2000, Goursat distributions were generalized in many works ([7], [12], [13], [14], [15], [20]). Special k-flags $(k \geq 2)$, which are considered to be extensions of Goursat flags, were defined in [7], [14], and [20] in several equivalent ways. All these approaches can be reduced to one transparent definition (see [1], [24]). A special k-flag of length s on a manifold M of dimension $(s+1)k+1$ is a sequence of distributions

$$D^s \subset D^{s-1} = [D^s, D^s] \subset \cdots \subset D^{j-1} = [D^j, D^j] \subset \cdots \subset D^0 = TM,$$

such that the respective dimensions of $D^s, D^{s-1}, \cdots, D^0$ are $k+1$, $2k+1, \cdots, (s+1)k+1$, for $j = 1, \cdots, s-1$, the Cauchy-characteristic sub-distribution $L(D^j)$ of D^j is included in D^{j+1} of constant corank one, $L(D^s) = 0$, and there exists a completely integrable sub-distribution $F \subset D^1$ of corank one in D^1. The integer k is called *width*.

The purpose of this work is to show that the problem of modelling car towing n trailers can be generalized to the problem of modelling kinematic problem for an “articulated arm” constituted of $(n+1)$ segments in $\mathbb{R}^{k+1}$, such that to this model is naturally associated a special k-flag.

In the following, an “articulated arm” of length $(n+1)$ is a series of $(n+1)$ segments $[M_i, M_{i+1}], i = 0, \cdots, n-1$, in $\mathbb{R}^{k+1}$, keeping a constant length l_i, and the articulation occurs at points M_i, for $i = 1, \cdots, n$.

It is proposed to study the kinematic evolution of the extremity M_0 under the constraint that the motion is controlled by the evolution of the segment $[M_n, M_{n+1}]$, and that the velocity of each point M_i,

$i = 0, \cdots, n,$ is colinear with the segment $[M_i, M_{i+1}]$ [1]. In this paper, we define precisely the kinematic model in angular coordinates of an articulated arm, and we construct the special multi-flag naturally associated to this model.

For $k = 1,$ an articulated arm of length n is a modelling problem of a car with n trailers: In this model, the car is symbolized by the segment $[M_n, M_{n+1}]$ (see [3]). When the number of trailers is large, this problem can be considered as an approximation of the "nonholonomic snake" in the plane (see [23], for instance). For $k > 1,$ we can also consider a "snake" in $\mathbb{R}^{k+1}$ (see [22] for a complete description). Again, an articulated arm of length n, for n large, can be considered as a discretization of a nonholonomic snake in $\mathbb{R}^{k+1}$. For instance, in $\mathbb{R}^3$, some problems of "towed cable" can model in such a way ([17], [23]).

In Section 2, we recall the classic context of the car with n trailers and its interpretation in terms of Goursat distribution. The articulated arm system is developed in Section 3 and also we show how to associate a special multi-flags to such a system in Cartesian coordinates. In Section 4, we gives a version of the kinematic evolution of an articulated arm in terms of angular coordinates, and we get a generalization of the classical model of the car with n trailers. The last two sections are devoted to the proofs of the results.

2. The Car with n Trailers

In this section, we will recall some fundamental results about the system of the car with n trailers and its relation with the Goursat distribution. All these results are now classical and can be found in a large number of papers as [2], [3], [11], [18], [26] and many others.

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[1] such a system is also studied in [10] and is called a "n-bar system".

2.1. Notations and equations

A car with n trailers is a configuration of $(n+1)$ trailers in the $\mathbb{R}^2$-plane, denoted by $M_0, M_1, \cdots, M_n$, and keeping a constant length between each two trailers. It is proposed to study the kinematic evolution of the trailer M_n with the constraint that the motion is controlled by the evolution of M_0, which symbolize the car. We will use the same representation as Fliess [2] and Sordalen [26], where the car is represented by two driving wheels connected by an axle. It is a kinematic problem with non integrable constraints (i.e., a nonholonomic system) due to the rolling without sliding of the wheels. The configuration space of the system is characterized by the two dimensional coordinates of M_n and $(n+1)$ angles, whereas there are only two inputs, namely, one tangential velocity and one angular velocity, which represent the action on the steering wheel and on the accelerator of the car. Consider the system of the car with n trailers and suppose that the distances R_r between the different trailers are all equal to 1. We choose as a reference point of a body M_{n-r} the midpoint m_r between the wheels; its coordinates are denoted by x_r and y_r in a given Cartesian frame of the plane; θ_r is the angle between the main axis of M_{n-r} and the x-axis of the frame. So, the set of all positions of the car with n trailers is included in a $3(n+1)$-dimensional space. This system is submitted to $2n$ holonomic links, which give, in the previous space, the $2n$ following equations:

$$\begin{aligned} x_r - x_{r-1} &= \cos\theta_{r-1}, \\ y_r - y_{r-1} &= \sin\theta_{r-1}. \end{aligned} \tag{1}$$

The configuration space of this problem is a submanifold of dimension $(n+3)$, which is parameterized by $q = (x_0, y_0, \theta_0, \ldots, \theta_n)$, where

- (x_0, y_0) are the coordinates of the last trailer M_n.

• θ_n is the orientation of the car (the trailer M_0) with respect to the x-axis.

• θ_r, $0 \leq r \leq n-1$, is the orientation of the trailer $(n-r)$ with respect to the x-axis.

The configuration space can thus be identified to $\mathbb{R}^2 \times (\mathbb{S}^1)^{n+1}$.

The velocity parameters are $\dot{x}_0, \dot{y}_0, \dot{\theta}_0, \cdots, \dot{\theta}_n$. There are only two inputs, namely, the "angular velocity" w_n and the "tangential velocity" v_n of the midpoint of the guiding wheels associated to the action of the car (see [3]).

Assume that the contacts between the wheels and the ground are pure rolling, it is then submitted to the classical nonholonomic links

$$\dot{x}_r \sin\theta_r - \dot{y}_r \cos\theta_r = 0. \tag{2}$$

There are $(n+1)$ kinematic equality constraints, one for each trailer. In order to establish these constraints, we can represent the points m_r, $r = 0, \cdots, n$, in the complex plane, i.e., $m_r = x_r + iy_r$. The geometric constraint between two consecutive trailers is written as

$$m_r = m_{r-1} + e^{i\theta_{r-1}}, \quad \text{for} \quad r \neq 0.$$

By induction, we have the following equation:

$$m_r = m_0 + \sum_{l=0}^{r-1} e^{i\theta_l}. \tag{3}$$

The kinematic constraint of M_{n-r} is

$$\dot{m}_r = \lambda_r e^{i\theta_r},$$

which is equivalent to

$$\mathcal{I}(e^{(-i\theta_r)}\dot{m}_r) = 0,$$

where $\mathcal{I}(z)$ denotes the imaginary part of z. Combining this characterization with the derivative of (3) and using the linearity of $\mathcal{I}$, we obtain the kinematic constraints

$$-\dot{x}_0 \sin\theta_r + \dot{y}_0 \cos\theta_r + \sum_{j=0}^{r-1} \dot{\theta}_j \cos(\theta_j - \theta_r) = 0, \quad r = 0, \cdots, n. \tag{4}$$

Combining $\dot{m}_r = \lambda_r e^{i\theta_r}$ with the derivative of

$$|m_{r+1} - m_r|^2 = 1,$$

we obtain

$$\lambda_r = \lambda_{r+1}(\cos\theta_{r+1} - \cos\theta_r),$$

and by induction

$$\lambda_r = \lambda_n \cos(\theta_n - \theta_{n-1}) \cdots \cos(\theta_{r+1} - \theta_r),$$

so

$$\dot{m}_r = \lambda_n \Big(\prod_{j=r+1}^{n} \cos(\theta_j - \theta_{j-1}) \Big) e^{i\theta_r},$$

where $\lambda_n = v_n$ is the tangential velocity of the car M_0.

The evolution of the system of car with n trailers can be given by the following controlled system with two controls v_n ("tangential velocity") and w_n ("normal velocity") of M_0:

$$\begin{cases} \dot{x}_0 = v_0 \cos\theta_0, \\ \dot{y}_0 = v_0 \sin\theta_0, \\ \dot{\theta}_0 = v_1 \sin(\theta_1 - \theta_0), \\ \qquad \cdots \\ \dot{\theta}_r = v_{r+1} \sin(\theta_{r+1} - \theta_r), \\ \qquad \cdots \\ \dot{\theta}_{n-1} = v_n \sin(\theta_n - \theta_{n-1}), \\ \dot{\theta}_n = w_n. \end{cases} \tag{5}$$

The "tangential velocity" v_r of the body M_{n-r} is given by

$$v_r = \prod_{j=r+1}^{n} \cos(\theta_j - \theta_{j-1}) v_n.$$

2.2. Goursat flag

Given a smooth distribution D on a manifold M, we will use the standard notation $[D, D]$ to denote the smooth distribution generated by the vector fields tangent to D and the Lie brackets $[X, Y]$, of any pair (X, Y) of vector fields tangent to D.

Definition 2.1. A Goursat flag of length s on a manifold M of dimension $l \geq s + 2$ is a sequence of distributions on M

$$D^s \subset D^{s-1} \subset \cdots \subset D^3 \subset D^2 \subset D^1 \subset D^0 = TM, \quad s \geq 2, \tag{F}$$

satisfying the following Goursat conditions:

(1) corang $D^i = i, \quad i = 1, 2, \cdots, s,$

(2) $D^{i-1} = [D^i, D^i], \quad i = 1, 2, \cdots, s.$ (G)

Each $D^i(p)$ is a subspace of T_pM of codimension i, for any point $p \in M$. It follows that $D^{i+1}(p)$ is a hyperplane in $D^i(p)$, for any $i = 0, 1, \cdots, s-1$ and $p \in M$.

Definition 2.2. We call any distribution D^i of corank $i \geq 2$ in a Goursat flag (F) a Goursat distribution.

To each flag (F) of Goursat distributions, we associate a flag of "Cauchy-characteristic" sub-distributions

$$L(D^s) \subset L(D^{s-1}) \subset \cdots \subset L(D^3) \subset L(D^2) \subset L(D^1), \tag{L}$$

where $L(D)$ is the sub-distribution of D generated by the set of vector fields X tangent to D such that $[X, Y]$ is tangent to D for all Y tangent to D. $L(D)$ is called the *Cauchy-characteristic distribution of D*.

Lemma 2.1 (Sandwich lemma [11]). *Let D be any Goursat distribution of corank $s \geq 2$ on a manifold M, and p be any point of M. Then*

$$L(D)(p) \subset L([D, D])(p) \subset D(p),$$

with $\dim L(D)(p) = \dim D(p) - 2, \quad \dim L([D, D])(p) = \dim D(p) - 1.$

It follows a relation between the Goursat flag and its flag of Cauchy-characteristic sub-distributions

$$\begin{array}{ccccccccccccc} & & D^s & \subset & D^{s-1} & \subset \cdots \subset & D^3 & \subset & D^2 & \subset & D^1 & \subset & D^0 \\ & & \cup & & \cup & & \cup & & \cup & & & & \\ L(D^s) & \subset & L(D^{s-1}) & \subset & L(D^{s-2}) & \subset \cdots \subset & L(D^2) & \subset & L(D^1). & & & & \end{array}$$

Each inclusion here is a codimension one inclusion of subbundles of the tangent bundle. $L(D^i)$ is an involutive regular distribution on M of codimension $i + 2$.

2.3. Goursat flag associated to the car with *n* trailers

Let

$$f_r^n = \prod_{j=r+1}^{n} \cos(\theta_j - \theta_{j-1}),$$

and $v_r = f_r^n v_n$ for $r = 1, \cdots, n-1$.

The motion of the system associated to the car is then characterized by the equation

$$\dot{q} = w_n X_n^1(q) + v_n X_n^2(q).$$

It is a controlled system with controls v_n and w_n, (v_n is the tangential velocity and w_n is the angular velocity as we have already seen at the beginning of the section). Each trajectory of the kinematic evolution of the car towing n trailers is an integral curve of the 2-distribution, on $\mathbb{R}^2 \times (\mathbb{S}^1)^{n+1}$, generated by

$$\begin{cases} X_n^1 = \dfrac{\partial}{\partial \theta_n}, \\ X_n^2 = \cos\theta_0 f_0^n \dfrac{\partial}{\partial x} + \sin\theta_0 f_0^n \dfrac{\partial}{\partial y} + \sin(\theta_1 - \theta_0) f_1^n \dfrac{\partial}{\partial \theta_0} + \cdots + \sin(\theta_n - \theta_{n-1}) \dfrac{\partial}{\partial \theta_{n-1}}. \end{cases}$$

The distribution generated by $\{X_1^n, X_2^n\}$, naturally associated to the system of the car with n trailers, is a Goursat distribution.

3. Articulated Arm

The purpose of this section is to construct a distribution Δ, of dimension $k+1$, naturally associated to an $(n+1)$ articulated arm, which generates a special k-flag of length $(n+1)$ on the configuration space $\mathcal{C} \equiv \mathbb{R}^{k+1} \times (\mathbb{S}^k)^{n+1}$. Moreover, the kinematic evolution of this arm is an integral curve of Δ_n. We begin by recalling the context of special multi-flag in the formalism of [1], [14].

3.1. Special multi-flags

A special k-flag of length s is a sequence

$$D = D^s \subset D^{s-1} \subset \cdots \subset D^j \subset \cdots \subset D^1 \subset D^0 = TM,$$

of distributions on a manifold M of dimension $(s+1)k+1$, which satisfies the following conditions:

(i) $D^{j-1} = [D^j, D^j]$.

(ii) $D^s, D^{s-1}, \cdots, D^j, \cdots, D^1, D^0$ are of respective ranks $k+1, 2k+1, \cdots, sk+1, (s+1)k+1$.

(iii) Each Cauchy characteristic sub-distribution $L(D^j)$ of D^j is a sub-distribution of constant corank one in each D^{j+1}, for $j = 1, \cdots, s-1$, and $L(D^s) = 0$.

(iv) There exists a completely integrable sub-distribution $F \subset D^1$ of corank one in D^1.

Remark 3.1. It should be remarked that the covariant sub-distribution $F \subset D^1$ is uniquely determined by D^1 itself. This covariant sub-distribution F is completely described in [1] and [24], where it is defined in terms of the annihilating Pfaffian system $(D^1)^{\perp} \subset T^*M$ ([7]). For a complete clarification on this fact, see [16].

Remark 3.2. In the following, we mean by a special multi-flag distribution all distribution generating a special multi-flag.

From the definition above, we obtain the following sandwich diagram:

$$\begin{array}{ccccccccc} D^s & \subset & D^{s-1} & \subset \cdots \subset & D^j & \subset \cdots \subset & D^1 & \subset D^0 & = TM \\ \cup & & \cup & \cdots & \cup & \cdots & \cup & & \\ L(D^{s-1}) & \subset & L(D^{s-2}) & \subset \cdots \subset & L(D^{j-1}) & \subset \cdots \subset & F. & & \end{array}$$

All vertical inclusions in this diagram are of codimension one, while all horizontal inclusions are of codimension k. The squares built by these inclusions can be perceived as certain sandwiches, i.e., each "subdiagram" number j indexed by the upper left vertices D^j

$$\begin{array}{ccc} D^j & \subset & D^{j-1} \\ \cup & & \cup \\ L(D^{j-1}) & \subset & L(D^{j-2}) \end{array}$$

is called *sandwich number j*.

We can read the length s of the special k-flag by adding one to the total number of sandwiches in the sandwich diagram.

Remark 3.3. In a sandwich number j, at each point $x \in M$, in the $(k+1)$ dimensional vector space $D^{j-1} / L(D^{j-1})(x)$, we can look for the relative position of the k dimensional subspace $L(D^{j-2}) / L(D^{j-1})(x)$ and the 1-dimensional subspace $D^j / L(D^{j-1})(x)$:

either $L(D^{j-2})/L(D^{j-1})(x) \oplus D^j / L(D^{j-1})(x) = D^{j-1} / L(D^{j-1})(x)$,

or $D^j / L(D^{j-1})(x) \subset L(D^{j-2})/L(D^{j-1})(x)$.

We say that $x \in M$ is a *regular point*, if the first situation is true in each sandwich number j, for $j = 1, \cdots, s$. Otherwise, x is called a *singular point*.

The set of singular points in the context of an articulated arm is studied in [25] and these results will be published in a future paper.

3.2. Special multi-flags and articulated arm

The space $(\mathbb{R}^{k+1})^{n+2}$, will be written as the product $\mathbb{R}_0^{k+1} \times \cdots \mathbb{R}_i^{k+1} \times \cdots \mathbb{R}_{n+1}^{k+1}$. Let $x_i = (x_i^1, \cdots, x_i^{k+1})$ be the canonical coordinates on the space $\mathbb{R}_i^{k+1}$, which is equipped with its canonical scalar product $<,>$. $(\mathbb{R}^{k+1})^{n+2}$ is equipped with its canonical scalar product too.

Consider an articulated arm of length $(n+1)$ denoted by $(M_0, \cdots, M_{n+1})$. In this paper, we assume that the distances l_i are all equal to 1. On $(\mathbb{R}^{k+1})^{n+2}$, consider the vector fields

$$\mathcal{Z}_i = \sum_{r=1}^{k+1} (x_{i+1}^r - x_i^r) \frac{\partial}{\partial x_i^r}, \quad \text{for } i = 0, \cdots, n. \tag{6}$$

From our previous assumptions (see Section 1), the kinematic evolution of the articulated arm is described by a controlled system

$$\dot{q} = \sum_{i=0}^{n} u_i \mathcal{Z}_i + \sum_{r=1}^{k+1} u_{n+r} \frac{\partial}{\partial x_{n+1}^r}, \tag{7}$$

with the following constraints:

$$\|x_i - x_{i+1}\| = 1, \text{ for } i = 0, \cdots, n \text{ (see [10], Chapter 2).}$$

Consider the map $\Psi_i(x_0, \cdots, x_{n+1}) = \|x_i - x_{i+1}\|^2 - 1$. Then, the configuration space $\mathcal{C}$ is the set

$$\{(x_0, \cdots, x_{n+1}), \text{ such that } \Psi_i(x_0, \cdots, x_{n+1}) = 0 \text{ for } i = 0, \cdots, n\}. \tag{8}$$

For $i = 0, \cdots, n,$ the vector field

$$\mathcal{N}_i = \sum_{r=1}^{k+1} (x_{i+1}^r - x_i^r)\Big[\frac{\partial}{\partial x_{i+1}^r} - \frac{\partial}{\partial x_i^r}\Big], \tag{9}$$

is proportional to the gradiant of Ψ_i. So, the tangent space $T_q\mathcal{C}$ is the subspace of $T_q(\mathbb{R}^{k+1})^{n+2}$, which is orthogonal to $\mathcal{N}_i(q)$ for $i = 0, \cdots, n.$

Denote by $\mathcal{E}$ the distribution generated by the vector fields

$$\{\mathcal{Z}_0, \cdots, \mathcal{Z}_n, \frac{\partial}{\partial x_{n+1}^1}, \cdots, \frac{\partial}{\partial x_{n+1}^{k+1}}\}.$$

Lemma 3.1. *Let* Δ *be the distribution on* $\mathcal{C}$ *defined by* $\Delta(q) = T_q\mathcal{C} \cap \mathcal{E}$. *Then* Δ *is a distribution of dimension* $k+1$ *generated by*

$$(x_{n+1}^r - x_n^r)\Big[\sum_{i=0}^{n} \prod_{j=i+1}^{n+1} A_j \mathcal{Z}_i\Big] + \frac{\partial}{\partial x_{n+1}^r}, \; \textit{for } r = 1, \cdots, k+1,$$

where $A_j(q) = < \mathcal{N}_j(q), \mathcal{N}_{j-1}(q) > = - < \mathcal{Z}_j(q), \mathcal{N}_{j-1}(q) >$ *for* $j = 1, \cdots, n$ *and* $A_{n+1} = 1$.

Proof. Any vector field X tangent to $\mathcal{E}$ can be written as

$$X = \sum_{i=0}^{n} \lambda_i \mathcal{Z}_i + \sum_{r=1}^{k+1} \mu_r \frac{\partial}{\partial x_{n+1}^r}.$$

On the other hand, on $\mathcal{C}$, a vector fields X is tangent to $\mathcal{C}$, if and only if X is orthogonal to the vector fields $\mathcal{N}_0, \cdots, \mathcal{N}_n$.

For $i = 0, \cdots, n-1$, each relation $< X, \mathcal{N}_i > = 0$ is reduced to $< \lambda_{i+1}\mathcal{Z}_{i+1} + \lambda_i \mathcal{Z}_i, \mathcal{N}_i > = 0$, which is equivalent to

$$\lambda_i = \lambda_{i+1} A_{i+1}. \tag{10}$$

Similarly, the relation $< X, \mathcal{N}_i > = 0$ induces

$$\lambda_n = \sum_{r=1}^{k+1} \mu_r (x_{n+1}^r - x_n^r), \tag{11}$$

and from (10) and (11), we get $\lambda_i = \prod_{j=i+1}^{n} A_i \lambda_n$, for $i = 0, \cdots, n-1$. □

The properties of Δ are summarized in the following result (see also [10], Chapter 2):

Theorem 3.1. *On* $\mathcal{C}$, *the distribution* Δ *satisfies the following properties*:

(1) Δ *is a distribution of rank* $k+1$.

(2) *The distribution* Δ *is a special k-flag on* $\mathcal{C}$ *of length* $(n+1)$.

The first part of Theorem 3.1 is a direct consequence of Lemma 3.1. Part (2) will be proved in Section 6 in terms of hyperspherical coordinates.

4. The Evolution of the Articulated Arm in a System of Angular Coordinates

Given an articulated arm $(M_0, \cdots, M_{n+1})$ in $\mathbb{R}^{k+1}$, we will show that the constraint controlled system (7) can be written in the same way as (5) in an adapted system of angular coordinates with $(k+1)$ controls, namely, v_n (the "normal" velocity of M_{n+1}) and the k components of the "tangential velocity" of M_{n+1} (Theorem 4.1).

4.1. Hyperspherical coordinates

The following map:

$$\Gamma(x_0, x_1, \cdots, x_i, \cdots, x_{n+1}) = (x_0, x_1 - x_0, \cdots, x_i - x_{i-1}, \cdots, x_{n+1} - x_n),$$

implies a global diffeomorphism of $(\mathbb{R}^{k+1})^{n+2}$ into itself and $\Gamma(\mathcal{C}) = \mathbb{R}^{k+1} \times (\mathbb{S}^k)^{n+1}$, where $\mathbb{S}^k$ is the canonical sphere in $\mathbb{R}^{k+1}$. In this representation, the canonical coordinates on $(\mathbb{R}^{k+1})^{n+2} = \Gamma((\mathbb{R}^{k+1})^{n+2})$ will be denoted by $(x_0, z_1, \cdots, z_i, \cdots, z_{n+1})$ so that Γ is given by $x_0 = x_0$ and $z_i = x_{i+1} - x_i$, for $i = 0, \cdots, n$. Via this global chart, each point $q = (x_0, x_1, \cdots, x_i, \cdots, x_{n+1}) \in \mathcal{C}$ can be identified with $(x_0, z_1, \cdots, z_i, \cdots, z_{n-1}) \in \mathbb{R}^{k+1} \times (\mathbb{S}^k)^{n+1}$, for $i = 0, \cdots, n$ and $\mathcal{C}$ can be identified with $\mathcal{S} = \mathbb{R}^{k+1} \times (\mathbb{S}^k)^{n+1}$.

We will put on $(\mathbb{S}^k)^{n+1}$ charts given by *hyperspherical coordinates*. We first recall some basic facts about this type of coordinates.

The *hyperspherical coordinates* in $\mathbb{R}^{k+1}$ are given by the relations

$$\begin{cases} z^1 = \rho\phi^1(\theta) = \rho \sin\theta^1 \cdots \sin\theta^{k-1} \sin\theta^k, \\ z^2 = \rho\phi^2(\theta) = \rho \sin\theta^1 \cdots \sin\theta^{k-1} \cos\theta^k, \\ z^3 = \rho\phi^3(\theta) = \rho \sin\theta^1 \cdots \sin\theta^{k-2} \cos\theta^{k-1}, \\ \quad \cdots \\ z^k = \rho\phi^k(\theta) = \rho \sin\theta^1 \cos\theta^2, \\ z^{k+1} = \rho\phi^{k+1}(\theta) = \rho\cos\theta^1, \end{cases} \tag{12}$$

with $\rho^2 = (z^1)^2 + \cdots + (z^{k+1})^2$, $0 \leq \theta^k \leq 2\pi$, and $0 \leq \theta^j \leq \pi$, for $1 \leq j \leq k - 1$.

We consider $\hat{\Phi}(\rho, \theta) = \rho\Phi(\theta) = z$, the application from $]0, +\infty[\times [0, \pi] \times \cdots \times [0, \pi] \times [0, 2\pi]$ to $\mathbb{R}^{k+1}$.

Remark 4.1. The previous expression uses the "geographical" version of hyperspherical coordinates. An another version, maybe more usual, can be obtained by taking $\frac{\pi}{2} - \theta^k$ instead of θ^k and then, permuting the functions sine and cosine in each formula. However, our choice is motivated by the following fact: The evolution of the articulated arm of length $(n + 1)$ written in a such chart, (see (16)) gives exactly the system (5) for $n = 1$.

The Jacobian matrix $D\hat{\Phi}$ of $\hat{\Phi}$ is:

$$\begin{pmatrix} \sin\theta^1\cdots\sin\theta^{k-1}\sin\theta^k & \rho\cos\theta^1\cdots\sin\theta^{k-1}\sin\theta^k & \cdots & \rho\sin\theta^1\cdots\sin\theta^{k-1}\cos\theta^k \\ \sin\theta^1\cdots\sin\theta^{k-1}\cos\theta^k & \rho\cos\theta^1\cdots\sin\theta^{k-1}\cos\theta^k & \cdots & -\rho\sin\theta^1\cdots\sin\theta^{k-1}\sin\theta^k \\ \cdots & \cdots & \cdots & \cdots \\ \cos\theta^1 & -\rho\sin\theta^1 & 0 & 0 \end{pmatrix}.$$

It is well known that $\det(D\hat{\Phi})(\rho,\theta) = (-1)^{[k+1/2]}(\rho)^k\prod_{i=1}^{k-1}(\sin\theta^{k-i})^i$.

It follows that $D\hat{\Phi}$ is invertible only for $0 \le \theta^k \le 2\pi$ and $0 < \theta^j < \pi$, for $j = 1, \cdots, k-1$.

In the sequence, we note $\mathcal{R}_{\hat{\Phi}} = \{\nu, \frac{\partial}{\partial\theta^1}, \cdots, \frac{\partial}{\partial\theta^k}\}$ the moving frame on $\hat{\Phi}(]0, +\infty[\times[0, \pi]\times\cdots\times]0, \pi[\times]0, 2\pi[)$, which is the image, by $D\hat{\Phi}$ of the canonical frame $\{\frac{\partial}{\partial t}, \frac{\partial}{\partial\theta^1}, \cdots, \frac{\partial}{\partial\theta^k}\}$.

Consider a point $z = \Phi(\theta) = \hat{\Phi}(1, \theta)$. We note that, in this case, we have

$$D\hat{\Phi} = \begin{pmatrix} \phi & \frac{\partial\phi}{\partial\theta^1} & \cdots & \frac{\partial\phi}{\partial\theta^k} \end{pmatrix},$$

where ϕ (resp., $\frac{\partial\phi}{\partial\theta^j}$) is the column vector of components $\{\phi^1, \cdots, \phi^{k+1}\}$ (resp., $\{\frac{\partial\phi^1}{\partial\theta^j}, \cdots, \frac{\partial\phi^{k+1}}{\partial\theta^j}\}$).

Moreover, these column vectors are pairwise orthogonal and we have

$$\|\phi\|^2 = \|\frac{\partial\phi}{\partial\theta^1}\|^2 = 1, \|\frac{\partial\phi}{\partial\theta^j}\|^2 = (\sin\theta^1\cdots\sin\theta^{j-1})^2, \text{ for } j = 2, \cdots, k.$$

The inverse of this matrix is then the transpose of the matrix

$$\begin{pmatrix} \phi & \dfrac{\dfrac{\partial\phi}{\partial\theta^1}}{\| \dfrac{\partial\phi}{\partial\theta^1} \|^2} & \cdots & \dfrac{\dfrac{\partial\phi}{\partial\theta^k}}{\| \dfrac{\partial\phi}{\partial\theta^k} \|^2} \end{pmatrix}. \tag{13}$$

This inverse exists only if $0 \leq \theta^k \leq 2\pi$ and $0 < \theta^j < \pi$ for $j = 1, \cdots, k-1$.

Consider a hyperspherical chart $y = \hat{\Phi}(\rho, \theta)$ and note $\{\nu, \frac{\partial}{\partial\theta^1}, \cdots, \frac{\partial}{\partial\theta^k}\}$ its associated frame field. Let $z' = \hat{\Phi}'(\rho', \theta')$ be another hyperspherical chart such that its domain intersects the domain of $\hat{\Phi}$. Note $\{\nu', \frac{\partial}{\partial\theta'^1}, \cdots, \frac{\partial}{\partial\theta'^k}\}$ the moving frame associated to it. So, we can write (on the intersection of these domains)

$$\nu' = A(\theta, \theta')\nu + \sum_{j=1}^{k} B^j(\theta, \theta') \frac{\partial}{\partial\theta^j}. \tag{14}$$

The components of these vectors are actually the components of the first column vector of the matrix $[D\hat{\Phi}]^{-1} \circ D\hat{\Phi}'$. At each point z such that $\rho = 1$, in view of (13), we get

$$\begin{cases} \bullet\ A(\theta, \theta') = \displaystyle\sum_{r=1}^{k+1} \phi^r \phi'^r, \\ \bullet\ B^1(\theta, \theta') = \displaystyle\sum_{r=1}^{k+1} \frac{\partial\phi^r}{\partial\theta^1} \phi'^r, \\ \bullet\ B^j(\theta, \theta') = \dfrac{1}{\| \dfrac{\partial\phi}{\partial\theta^j} \|^2} \displaystyle\sum_{r=1}^{k+1} \frac{\partial\phi^r}{\partial\theta^j} \phi'^r. \end{cases} \tag{15}$$

4.2. The evolution of the articulated arm on $\mathcal{S}$

Coming back to $\mathcal{S} = \mathbb{R}^{k+1} \times (\mathbb{S}^k)^{n+1}$, which is considered as a subset in $(\mathbb{R}^{k+1})^{n+2}$, let $\mathbb{S}_i$, $i = 0, \cdots, n$, be the canonical sphere in $\mathbb{R}^{k+1}_{i+1}$.

Recall that the canonical coordinates on $\mathbb{R}_i^{k+1}$ are denoted by $z_i = (z_i^1, \cdots, z_i^r, \cdots, z_i^{k+1})$. Given a point α in the sphere $\mathbb{S}_i$, there exists a hyperspherical chart $z_{i+1} = \hat{\Phi}_i(\rho_i, \theta_i) = \rho_i \Phi_i(\theta_i^1, \cdots, \theta_i^k)$ defined for $0 \leq \theta_i^k \leq 2\pi$ and $0 < \theta_i^j < \pi$, $j = 1, \cdots, k-1$, where $\Phi_i(0, \cdots, 0) = \alpha$. So, for a given point $q = (x_0, z_1, \cdots, z_i, \cdots, z_{n+1}) \in \mathcal{S}$, we get a chart $(Id - x_0, \hat{\Phi}_0, \cdots, \hat{\Phi}_1, \cdots, \hat{\Phi}_n)$ centered at q, such that its restriction to $\rho_i = 1$, $i = 0, \cdots, n$, induces a chart of $\mathcal{S}$ (centered at q). For $i = 0, \cdots, n$, in a neighborhood of each $z_{i+1} \in \mathbb{R}_{i+1}^{k+1}$, we consider the moving frame $\mathcal{R}_i = \{\nu_i, \frac{\partial}{\partial\theta_i^1}, \cdots, \frac{\partial}{\partial\theta_i^k}\} = \mathcal{R}_{\hat{\Phi}_i}$ (with notations introduced above).

Remark 4.2. Given $q = (x_0, \cdots, x_{n+1}) \in \mathcal{C}$, for $i = 0, \cdots, n$, denote by $\widetilde{\mathbb{S}}_i$ the sphere in $\mathbb{R}^{k+1}$ of center x_i and radius 1. One can put on $\mathbb{R}^{k+1}$ the hyperspherical coordinates $y_i = \hat{\Phi}_i(\rho_i, \theta_i) + x_i$. As x_{i+1} belongs to $\widetilde{\mathbb{S}}_i$, on a neighborhood of x_{i+1}, we also have the following moving frame (again denoted $\mathcal{R}_i$):

$$\mathcal{R}_i = \{\nu_i, \frac{\partial}{\partial\theta_i^1}, \cdots, \frac{\partial}{\partial\theta_i^k}\}.$$

Note that on x_{i+1}, the outward normal unit vector of $\widetilde{\mathbb{S}}_i$ is $\nu_i(x_{i+1})$ and $\{\frac{\partial}{\partial\theta_i^1}, \cdots, \frac{\partial}{\partial\theta_i^k}\}$ is a basis $T_{x_{i+1}}\widetilde{\mathbb{S}}_i$.

Notations 4.1. We define on $\mathcal{S}$:

(1) $A_i = \sum_{r=1}^{k+1} \phi_{i-1}^r \phi_i^r$, for $i = 1, \cdots, n$ and $A_{n+1} = 1$;

(2) $Z_0 = \sum_{r=1}^{k+1} \phi_0^r \frac{\partial}{\partial x^r}$;

(3) $Z_i = \sum_{j=1}^{k} B_i^j \frac{\partial}{\partial \theta_{i-1}^j}$, for $i = 1, \cdots, n$;

with

- $B_i^1 = \sum_{r=1}^{k+1} \frac{\partial \phi_{i-1}^r}{\partial \theta_{i-1}^1} \phi_i^r$, for $i = 1, \cdots, n$,

- $B_i^j = \frac{1}{\| \frac{\partial \phi_{i-1}}{\partial \theta_{i-1}^j} \|^2} \sum_{r=1}^{k+1} \frac{\partial \phi_{i-1}^r}{\partial \theta_{i-1}^j} \phi_i^r$, for $i = 1, \cdots, n$ and $j = 2, \cdots, k$,

(4) $X_m^i = \frac{\partial}{\partial \theta_m^i}$, for $i = 1, \cdots, k$ and $m = 0, \cdots, n$;

(5) $X_m^0 = \sum_{i=0}^{m} f_m^i Z_i$, for $m = 0, \cdots, n$;

with $f_m^r = \prod_{j=r+1}^{m} A_j$, for $r = 0, \cdots, m-1$, and $f_m^m = 1$, for $m = 0, \cdots, n$.

(6) Δ_n the distribution generated by $\{X_n^0, X_n^1, \cdots, X_n^k\}$ (with previous notations).

Remark 4.3. For $i = 0, \cdots, n$, consider $[\Phi_i]$ the column matrix of components $(\phi_i^1, \cdots, \phi_i^{k+1})$ and $[D\Phi_i]^{-1}$ the matrix composed by the last k rows of the Jacobian matrix of the application $(\Phi_i)^{-1}$. Finally, denote by $[\frac{\partial}{\partial \theta_i}]$ (resp., $[\frac{\partial}{\partial x}]$) the column matrix of components $(\frac{\partial}{\partial \theta_i^1}, \cdots, \frac{\partial}{\partial \theta_i^k})$, (resp., $(\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^{k+1}})$), for $i = 0, \cdots, n$. So, we can write

$$Z_0 = {}^t \left[\frac{\partial}{\partial x}\right][\Phi_0] \text{ and } Z_i = {}^t \left[\frac{\partial}{\partial \theta_{i-1}}\right][D\Phi_{i-1}]^{-1}[\Phi_i], \text{ for } i = 1, \cdots, n.$$

With these notations, we have the following result:

Theorem 4.1. (1) *On* $\mathcal{S}$, *the distribution* Δ_n *is the image by* Γ *of the distribution* Δ, *where* $\Gamma : \mathcal{C} \to \mathcal{S}$ *is the diffeomorphism defined at the beginning of Subsection* 4.1.

(2) *The evolution of the articulated arm of length* $(n+1)$ *is described in a chart, by the following controlled system with* $k+1$ *controls*:

$$\begin{cases} \dot{x}^1 = v_0 \phi_0^1, \\ \dot{x}^2 = v_0 \phi_0^2, \\ \quad \dots \\ \dot{x}^{k+1} = v_0 \phi_0^{k+1}, \\ \dot{\theta}_0^1 = v_1 B_1^1, \\ \quad \dots \\ \dot{\theta}_0^k = v_1 B_1^k, \\ \quad \dots \\ \dot{\theta}_i^1 = v_{i+1} B_{i+1}^1, \\ \quad \dots \\ \dot{\theta}_i^k = v_{i+1} B_{i+1}^k, \\ \quad \dots \\ \dot{\theta}_{n-1}^1 = v_n B_n^1, \\ \quad \dots \\ \dot{\theta}_{n-1}^k = v_n B_n^k, \\ \dot{\theta}_n^1 = v_{\theta_n^1}, \\ \quad \dots \\ \dot{\theta}_n^k = v_{\theta_n^k}, \end{cases} \tag{16}$$

where $v_i = v_n \prod_{r=i+1}^{n} A_r$ *and* $(v_{\theta_n^1}, \cdots, v_{\theta_n^k}, v_n)$ *are the* $(k+1)$ *controls of the system* (16).

Moreover, according to Remark 4.2 (*or Lemma* 5.2), *we have*

• $(v_{\theta_n^1}, \cdots, v_{\theta_n^k})$ *are the "tangential components" of the velocity of* M_{n+1}, *namely, the components, in the canonical basis of* $T_{x_{n+1}}\widetilde{\mathbb{S}}_n$, *of the orthogonal projection of the velocity of* M_{n+1};

• v_i *is the normal velocity of* M_i *for all* $i = 1, \cdots, n+1$, *namely, the components of the orthogonal projection of the velocity of* M_i *on the direction generated by* $\nu_{i-1}(x_i)$.

Remark 4.4. Equations of system (16), for $k = 1$, are exactly (with the same notations) the classical modelling of the car with n trailers ([2], [3], [8], [19], [26]).

Remark 4.5. According to Remark 3.3, for $k \geq 2$, a point $q = (x_0, z_1, \cdots, z_{n+1})$ is singular, if and only if there exists an index $0 \leq i \leq n$ such that $A_i(q) = 0$, which is equivalent to $[M_{i-1}, M_i]$ and $[M_i, M_{i+1}]$ are orthogonal in M_i. In this situation, all the "normal velocity" of M_j are zero, for $j > i$. The set of such points is studied in [25].

5. Proof of Theorem 4.1

5.1. Preliminaries and proof of part (1)

We will use the notations introduced at the beginning of Subsection 4.1. We first have the following lemma:

Lemma 5.1. *Consider, on* $\mathcal{C}$, *the natural decomposition:*

$$[T(\mathbb{R}^{k+1})^{n+2}]_{|\mathcal{C}} = T\mathcal{C} \oplus T\mathcal{C}^{\perp},$$

where $T\mathcal{C}^{\perp}$ *is the orthogonal of* $T\mathcal{C}$. *Denote by* $\prod$ *the orthogonal projection of* $[T(\mathbb{R}^{k+1})^{n+2}]_{|\mathcal{C}}$ *on* $T\mathcal{C}$.

(1) *The family of vector field* $\{\prod(\frac{\partial}{\partial x_{i+1}^r})(p),\ r = 1, \cdots, k+1\}$ *generates the tangent space at* $p \in \mathcal{C}$ *of the sphere of equation* $\Psi_i = 0,\ i = 0, \cdots, n.$

(2) *On* $\mathcal{C}$, *let* $\mathcal{L}$ *be the involutive distribution, whose leafs are defined by* $\Psi_n = 0$. *The distribution* Δ, *is generated by* $\mathcal{L}$ *and the vector field* $X = [\sum_{i=0}^{n}\prod_{j=i+1}^{n+1} A_j \prod(\mathcal{Z}_i)]$.

Moreover, $Z_i = \prod(\mathcal{Z}_i)$ *is tangent to the sphere of equation* $\Psi_{i-1} = 0$, $i = 1, \cdots, n.$

Proof. At first, for any $p \in \mathcal{C}$, note that

$$\Delta_p = \{v \in \mathcal{E}_p \text{ such that } < v, \nu >= 0,\ \forall \nu \in T_p\mathcal{C}^{\perp}\},$$

so Δ is nothing but $\prod(\mathcal{E})$.

Note that, at each point $p = (x_0, \cdots, x_{n+1}) \in \mathcal{C}$, the vector $\frac{\partial}{\partial x_{i+1}^r}(p)$ satisfies

$$dx_j^s(\frac{\partial}{\partial x_{i+1}^r}) = 0, \text{ for any } r, s = 1, \cdots, k+1,$$

and

$$j = 0, \cdots, i, i+2, \cdots, n+1.$$

So, the family of vector field $\{\prod(\frac{\partial}{\partial x_{i+1}^r})(p),\ r = 1, \cdots, k+1\}$ generates the tangent space at p of the sphere of equation $\Psi_i = 0$, which proves (1).

The integrable distribution $\mathcal{F}$, on $(\mathbb{R}^{k+1})^{n+2}$, generated by $\{\frac{\partial}{\partial x_{n+1}^r},$ $r = 1, \cdots, k+1\}$, is contained in $\mathcal{E}$, and the distribution $\mathcal{L} = \prod(\mathcal{E})$ induced on $\mathcal{C}$ by $\mathcal{F}$ is also integrable and, of course is contained in Δ. In particular, $\mathcal{L}$ is generated by $\{\Pi(\frac{\partial}{\partial x_{n+1}^r}), r = 1, \cdots, k+1\}$. From Lemma 3.1, the vector field $Y = [\sum_{i=0}^{n} \prod_{j=i+1}^{n+1} A_j \mathcal{Z}_i] + [\sum_{r=1}^{k+1} (x_{n+1}^r - x_n^r) \frac{\partial}{\partial x_{n+1}^r}]$ is tangent to $\mathcal{C}$. But Δ is a distribution of constant rank $k+1$ and $\mathcal{L}$ is an (integrable) sub-distribution of rank k. It follows that Δ is generated by $\mathcal{L}$ and Y. On the other hand, as Y is tangent to $\mathcal{C}$, we have

$$Y = \Pi(Y) = [\sum_{i=0}^{n} \prod_{j=i+1}^{n+1} A_j\, \Pi[\mathcal{Z}_i]] + \Pi[\sum_{r=1}^{k+1} (x_{n+1}^r - x_n^r) \frac{\partial}{\partial x_{n+1}^r}].$$

As $\Pi[\sum_{r=1}^{k+1} (x_{n+1}^r - x_n^r) \frac{\partial}{\partial x_{n+1}^r}]$ is tangent to $\mathcal{L}$, it follows that Δ is generated by $\mathcal{L}$ and

$$X = \sum_{i=0}^{n} \prod_{j=i+1}^{n+1} A_j\, \Pi[\mathcal{Z}_i].$$

On the other hand, we have

$$Z_i = \Pi(\mathcal{Z}_i) = \sum_{r=1}^{k+1} (x_{i+1}^r - x_i^r)\, \Pi(\frac{\partial}{\partial x_i^r}). \tag{17}$$

So, from (1), Z_i is tangent to the sphere of equation $\Psi_{i-1} = 0$.

□

Proof of Theorem 4.1 part (1)

From Lemma 5.1, in a hyperspherical chart on $\mathcal{S}$, the distribution $\mathcal{K} = \Gamma(\mathcal{L})$ is generated by $\{X_n^i = \frac{\partial}{\partial \theta_n^i}, i = 1, \cdots, k\}$. Moreover, the distribution $\Delta_n = \Gamma_*(\Delta)$ is generated by

$\{X_n^i = \frac{\partial}{\partial \theta_n^i}, i = 1, \cdots, k\}$ and $X_n^0 = \Gamma_*(X)$ (with notations of Lemma 5.1).

It remains to show that the vectors field Z_i, $i = 0, \cdots, n$ can be written as in Notation 4.1 (2) and (3).

Fix $q = (x_0, z_1, \cdots, z_{n+1}) \in \mathcal{S}$ and again set $x_i = x_{i-1} + z_i$, for $i = 1, \cdots, n+1$ and put on $\mathcal{S} \subset (\mathbb{R}^{k+1})^{n+2}$ coordinates chart $(Id - x_0, \hat{\Phi}_0, \cdots, \hat{\Phi}_1, \cdots, \hat{\Phi}_n)$ centered at q, such that its restriction to $\rho_i = 1$, $i = 0, \cdots, n$, induces a chart of $\mathcal{S}$ (centered at q). In this chart, we have $z_i = \Phi_{i-1}(\theta_i)$ for $i = 1, \cdots, n+1$.

So, from (17), using (15), for hyperspherical coordinates $\hat{\Phi}_i(\rho_i, \theta_i)$ and $\hat{\Phi}_{i-1}(\rho_{i-1}, \theta_{i-1})$, we obtain the announced expression of Z_i of in hyperspherical coordinates.

$\square$

5.2. Proof of part (2)

Taking in account part (1) of Theorem 4.1, the kinematic evolution of the articulated arm is a controlled system on $\mathcal{S}$, which is exactly (16). However, for the completeness of the proof of this result, we must prove the interpretation of the control in terms of the component of the velocity of M_i, $i = 0, \cdots, n+1$.

Consider a point $q = (x_0, z_1, \cdots, z_{n+1}) \in \mathcal{S}$, and we set again $x_{i+1} = x_i + z_{i+1}$, for $i = 0, \cdots, n$. According to Remark 4.2, the tangent space $T_q\mathcal{S}$ can be identified with

$$T_{x_0}\mathbb{R}_0^{k+1} \times T_{x_1}\widetilde{\mathbb{S}}_0 \times \cdots \times T_{x_{n+1}}\widetilde{\mathbb{S}}_n \equiv T_{x_0}\mathbb{R}_0^{k+1} \oplus T_{x_1}\widetilde{\mathbb{S}}_0 \oplus \cdots \oplus T_{x_{n+1}}\widetilde{\mathbb{S}}_n, \quad (18)$$

which is a subspace of the tangent space $T_q(\mathbb{R}^{k+1})^{n+2}$, which can be identified with

$$T_{x_0}\mathbb{R}_0^{k+1} \times T_{x_1}\mathbb{R}^{k+1} \times \cdots \times T_{x_{n+1}}\mathbb{R}^{k+1} \equiv T_{x_0}\mathbb{R}_0^{k+1} \oplus T_{x_1}\mathbb{R}^{k+1} \oplus \cdots \oplus T_{x_{n+1}}\mathbb{R}^{k+1}.$$

Remark 5.1. Each pair (x_i, z_i), $i = 1, \cdots, n+1$, can be considered as a vector of $T_{x_i}\mathbb{R}^{k+1}$, which is equal to $\nu_{i-1}(x_i)$. On the other hand, for $i = 0, \cdots, n$, x_i belongs to the sphere $\widetilde{\mathbb{S}}'_i$ centered at $-x_{i+1}$ in $\mathbb{R}^{k+1}$. Given the hyperspherical coordinate $y'_i = \hat{\Phi}_i(\rho_i, \theta_i) - x_i$. On a neighborhood of x_i, we also have the following moving frame (again denoted by $\mathcal{R}_i$):

$$\mathcal{R}_i = \{\nu_i, \frac{\partial}{\partial\theta_i^1}, \cdots, \frac{\partial}{\partial\theta_i^k}\}.$$

In these conditions, (x_i, z_{i+1}) can be identified with $\nu_i(x_i)$.

Now, in $\mathbb{R}^{k+1}$, given any trajectory of an articulated arm, we have

$$x_i = x_{i-1} + z_i, \quad \text{for } i = 1, \cdots, n+1, \tag{19}$$

from our assumptions (see Section 1), we also have

$$\dot{x}_{i-1} = v_{i-1}z_i, \quad \text{for } i = 1, \cdots, n+1, \tag{20}$$

for some $v_{i-1} \in \mathbb{R}$.

In view of (20), the derivative of (19) can be written as

$$\dot{x}_i = \dot{x}_{i-1} + \dot{z}_i = v_{i-1}z_i + \dot{z}_i, \quad i = 1, \cdots, n+1. \tag{21}$$

The pair $(x_i, \dot{x}_i)$, $i = 1, \cdots, n$, can be considered as vector in $T_{x_i}\mathbb{R}^{k+1}$. Since $x_i \in \widetilde{\mathbb{S}}_{i-1}$, taking in account Remark 5.1, using (21), it follows that the pair $(x_i, \dot{x}_i) \in T_{x_i}\mathbb{R}^{k+1}$ can be written

$$(x_i, \dot{x}_i) = v_{i-1}\nu_{i-1}(x_i) + (x_i, w_i), \tag{22}$$

where (x_i, w_i) belongs to $T_{x_i}\widetilde{\mathbb{S}}_{i-1}$.

From Remark 5.1, we have the orthogonal decomposition

$$\nu_i(x_i) = < z_{i+1}, z_i > \nu_{i-1}(x_i) + \widetilde{Z}_i(x_i), \quad i = 0, \cdots, n. \tag{23}$$

Taking in account our assumptions and the identification $(x_i, z_{i+1}) \equiv \nu_i(x_i)$ (see Remark 5.1), we also have

$$(x_i, \dot{x}_i) = \upsilon_i(x_i, z_{i+1}) = \upsilon_i\nu_i(x_i), \tag{24}$$

for some $\upsilon_i \in \mathbb{R}$.

From (23) and (24), we obtain

$$(x_i, \dot{x}_i) = \upsilon_i < z_{i+1}, z_i > \nu_{i-1}(x_i) + \upsilon_i\widetilde{Z}_i(x_i). \tag{25}$$

Finally, comparing (22) and (25), we get

$$\upsilon_{i-1} = \upsilon_i < z_{i+1}, z_i > \text{ and } (x_i, w_i) = \upsilon_i\widetilde{Z}_i(x_i), \quad \text{for } i = 1, \cdots, n. \tag{26}$$

For the pair $(x_{n+1}, \dot{x}_{n+1})$ considered as a vector in $T_{x_{n+1}}\mathbb{R}^{k+1}$, we have the orthogonal decomposition

$$(x_{n+1}, \dot{x}_{n+1}) = \upsilon_n\nu_n(x_{n+1}) + (x_{n+1}, w_{n+1}), \tag{27}$$

where $(x_{n+1}, w_{n+1}) \in T_{x_{n+1}}\widetilde{\mathbb{S}}_n$.

So, we have proved the following lemma:

Lemma 5.2. (1) *For* $i = 1, \cdots, n-1,$ *we have*

$$(x_i, \dot{x}_i) = \upsilon_{i-1}\nu_{i-1}(x_i) + \upsilon_i\widetilde{Z}_i(x_i),$$

where $Z_i(x_i) = \nu_i(x_i) - A_i(q)\nu_{i-1}(x_i)$ *and* $A_i(q) = < z_{i+1}, z_i >$.

(2) $(x_{n+1}, \dot{x}_{n+1}) = \upsilon_n\nu_n(x_{n+1}) + (x_{n+1}, w_{n+1}),$

for some $(x_{n+1}, w_{n+1}) \in T_{x_{n+1}}\widetilde{\mathbb{S}}_n$.

(3) $\upsilon_i = \Big(\prod_{j=i+1}^{n} A_j\Big)\upsilon_n, \quad$ *for* $i = 0, \cdots, n-1.$

In this context, υ_i (*resp.,* w_i) *is called the normal* (*resp., tangential*) *velocity of* M_i.

Remark 5.2. (1) Via the identification (18) and (19), $\widetilde{Z}_i(x_i)$ can be identified with $Z_i(z_i)$ on $\mathcal{S}$ given in Notation 4.1 (2) and (3).

(2) The function $A_i(x_0, z_1, \cdots, z_{n+1})$ is exactly $A_i(x_0, x_1, \cdots, x_{n+1})$, as defined in Lemma 3.1.

Sketch of the proof of Theorem 4.1 part (2)

From Lemma 5.2, the following decomposition occurs:

$$(x_{n+1}, \dot{x}_{n+1}) = v_n \nu_n(x_{n+1}) + (x_{n+1}, w_{n+1}),$$

where $(x_{n+1}, w_{n+1}) \in T_{x_{n+1}} \widetilde{\mathbb{S}}_n$.

Let $(v_{\theta_n^1}, \cdots, v_{\theta_n^k})$ be the components of (x_{n+1}, w_{n+1}) relative to the basis $\{ \frac{\partial}{\partial \theta_n^1}, \cdots, \frac{\partial}{\partial \theta_n^k} \}$ of $T_{x_{n+1}} \widetilde{\mathbb{S}}_n$, with $\dot{\theta}_n^j = v_{\theta_n^j}$, for $j = 1, \cdots, k$.

From (23), we deduce $\nu_n(x_n) = < z_{n+1}, z_n > \nu_{n-1}(x_n) + Z_n(x_n)$, and from (26) and (27), we have $(x_n, w_n) = v_n Z_n(x_n)$.

Then by applying (14), we obtain

$$\nu_n(x_n) = A(\theta_{n-1}, \theta_n) \nu_{n-1} x_n + \sum_{j=1}^{k} B^j(\theta_{n-1}, \theta_n) \frac{\partial}{\partial \theta_{n-1}^j},$$

with

$$A_n(q) = A(\theta_{n-1}, \theta_n) \text{ and } Z_n = \sum_{j=1}^{k} B^j(\theta_{n-1}, \theta_n) \frac{\partial}{\partial \theta_{n-1}^j}.$$

Moreover, $\dot{\theta}_{n-1}^j = v_n B^j(\theta_{n-1}, \theta_n)$, for $j = 1, \cdots, k$. In particular, $(v_{\theta_n^1}, \cdots, v_{\theta_n^k})$ are the components of the tangential velocity of M_{n+1} and v_n is the normal component of the velocity of M_{n+1}.

Note that since $z_{n+1} = \Phi_n(\theta_n)$ and $z_n = \Phi_{n-1}(\theta_{n-1})$, the value $A_n(q)$ defined in Lemma 5.2 is exactly $A(\theta_{n-1}, \theta_n)$ (see (15)).

The end of the proof of part (2) can be done by induction, using the same arguments as before, and taking in account (23), (25), (26), and Lemma 5.2 or is a direct consequence of part (1).

□

6. Proof of Theorem 3.1

We will see that the distribution Δ_n generates actually a special k-flag of length $(n+1)$ on a $k(n+2)+1$ dimensional manifold. Let us introduce the following notations:

- Δ_m is the distribution generated by $\{X_m^0, X_m^1, \cdots, X_m^k\}$, for $m = 1, \cdots, n$;
- D^{m+1} is the distribution generated by X_m^0 and $\{X_j^1, \cdots, X_j^k, m \leq j \leq n\}$, for $m = 0, \cdots, n$;
- $D^0 = TM$;
- E^{m+1} is the distribution generated by $\{X_j^1, \cdots, X_j^k, m \leq j \leq n\}$, for $m = 0, \cdots, n$.

Proposition 6.1. Δ_n *is a special k-flag distribution. More precisely, it satisfies the following properties*:

(1) *For* $m = 1, \cdots, n+1$, *the distributions* D^m *and* E^m *are of respective constant dimensions* $(n-m+2)k+1$ *and* $(n-m+2)k$;

(2) *for* $m = 1, \cdots, n+1$, E^m *is an involutive sub-distribution of* D^m *of codimension* 1. *Moreover*, $[E^{m+1}, D^{m+1}] \subset D^m$, *for* $m = 1, \cdots, n$. *Actually*, E^{m+1} *is the "Cauchy-characteristic distribution" of* D^m *for* $m = 1, \cdots, n$ ([13], [14]);

(3) $[D^{m+1}, D^{m+1}] = D^m$, *for all* $m = 0, \cdots, n$;

(4) $$\begin{array}{ccccccccc} \Delta_n = D^{n+1} & \subset & \cdots & \subset & D^m & \subset \cdots \subset & D^1 & \subset D^0 = TM \\ \cup & & \cdots & & \cup & \cdots & \cup & \\ E^{n+1} & \subset & \cdots & \subset & E^m & \subset \cdots \subset & E^1. & \end{array}$$

Proof of Proposition 6.1

It is sufficient to show the property (4). The inclusions $[E^{m+1}, D^{m+1}] \subset D^m$ for $m = n, \cdots, 0$, are an easy consequence of (4) and the properties (1), (2), and (3) are always true, according to the definition of spaces E^m, D^m, and Δ_m.

Denote by Δ_0 the distribution generated by $\{\frac{\partial}{\partial x_0^1}, \cdots, \frac{\partial}{\partial x_0^{k+1}}\}$.

For all $m = 1, \cdots, n+1$, we have

$$D^m = E^{m+1} \oplus \Delta_{m-1} = D^{m+1} + \Delta_{m-1}.$$

$[D^{m+1}, D^{m+1}]$ contains the space generated by D^{m+1} and the Lie brackets $[X_m^i, X_m^0]$, for $i = 1, \cdots, k$. We will show by induction that, in fact, they are generating $[D^{m+1}, D^{m+1}]$ modulo D^{m+1}.

For all $m = n, \cdots, 0$, we have $X_m^0 = A_m X_{m-1}^0 + Z_m$. It results from the definition of A_i, Z_i, and X_m^0 that $[X_m^i, X_{m-1}^0] = 0$. So, we have

$$[X_m^i, X_m^0] = X_m^i(A_m)X_{m-1}^0 + [X_m^i, Z_m].$$

For $j = 1, \cdots, k+1$, consider the vector fields

$$Y_{m-1}^j = \phi_{m-1}^j X_{m-1}^0 + \sum_{r=1}^{k} \frac{1}{\| \frac{\partial \phi_{m-1}}{\partial \theta_{m-1}^r} \|^2} \frac{\partial \phi_{m-1}^j}{\partial \theta_{m-1}^r} X_{m-1}^r.$$

If we set $\hat{\Phi}_{m-1}(\rho, \theta_{m-1}) = \rho\Phi_{m-1}(\theta_{m-1})$, then we have the relation

$$[Y_{m-1}] = [D\hat{\Phi}_{m-1}]^{-1}[X_{m-1}],$$

where the vectors column $[Y_{m-1}]$ and $[X_{m-1}]$ have $\{Y_{m-1}^1, \cdots, Y_{m-1}^{k+1}\}$ and $\{X_{m-1}^0, \cdots, X_{m-1}^k\}$ as components, respectively. It results that $\{Y_{m-1}^1, \cdots, Y_{m-1}^{k+1}\}$ is a basis of Δ_{m-1}.

For $m = 0, \cdots, n,$ we note $[D\hat{\Phi}_m]$ the Jacobian matrix of $\hat{\Phi}_m(\rho, \theta_m) = \rho\Phi_m(\theta_m)$.

The following decompositions occur:

$$X_m^0 = \sum_{j=1}^{k+1} \phi_m^j Y_{m-1}^j,$$

$$[X_m^i, X_m^0] = \sum_{j=1}^{k+1} \frac{\partial \phi_m^j}{\partial \theta_m^i} Y_{m-1}^j, \text{ for all } m = 1, \cdots, n.$$

By similar way, for $D\hat{\Phi}_m$, we can show that the family of vector fields

$$\{X_m^0, [X_m^1, X_m^0], \cdots, [X_m^k, X_m^0]\},$$

is also a basis of Δ_{m-1}. This result is also true for $m = 0$.

Since $D^{m+1} = E^{m+2} \oplus \Delta_m$, the space $[D^{m+1}, D^{m+1}]$ contains E^{m+2}, all vectors $X_m^0, X_m^1, \cdots, X_m^k$ and the Lie brackets $[X_m^1, X_m^0], \cdots, [X_m^k, X_m^0]$. Also, all the Lie brackets $[X_r^j, X_m^0]$ are zero for $r = m+1, \cdots, n$ and $j = 1, \cdots, k,$ since X_m^0 does not depend on variables θ_r^j, for $r = m+1, \cdots, n$ and $j = 1, \cdots, k.$ The other Lie brackets $[X_r^j, X_m^i]$ are zero for $r = m+1, \cdots, n$ and $i, j = 1, \cdots, k.$

Since $\{X_m^0, [X_m^1, X_m^0], \cdots, [X_m^k, X_m^0]\}$ is a basis of Δ_{m-1}, then we have

$$[D^{m+1}, D^{m+1}] = E^{m+1} \oplus \Delta_{m-1} = D^m,$$

which completes the proof of Proposition 6.1 and Theorem 3.1.

□

Comment 6.1. Given two integers p and m such that $1 \leq p < m \leq n$, we can look for the motion of a "sub-induced arm", which consists of segments of the original arm between M_{p-1} and M_{m+1} included. We can then study the motion of M_{p-1} as the motion of the extremity of this sub-arm for the motion commanded by the segment $[M_m, M_{m+1}]$. We put $h = m - p + 1$, and we write $\prod_{p,m}$ for the canonical projection from $\mathbb{R}^{k+1} \times (\mathbb{S}^k)^{n+1}$ onto $\mathbb{R}^{k+1} \times (\mathbb{S}^k)^{h+1}$ defined as $\prod_{p,m}(x, z_1, \cdots, z_{n+1}) = (x_{p-1}, z_{p-1}, z_p, \cdots, z_m)$, where x_{p-1} are the Cartesian coordinates of M_{p-1}.

The evolution of the extremity M_{p-1} of this articulated sub-arm, controlled by the movement of $[M_m, M_{m+1}]$, is a solution of the following differential system (with notations of Theorem 4.1):

$$\begin{cases} \dot{x}_{p-1}^1 = v_{p-1}\phi_{p-1}^1, \\ \dot{x}_{p-1}^2 = v_{p-1}\phi_{p-1}^2, \\ \quad \cdots \\ \dot{x}_{p-1}^{k+1} = v_{p-1}\phi_{p-1}^{k+1}, \\ \dot{\theta}_{p-1}^1 = v_p B_p^1, \\ \quad \cdots \\ \dot{\theta}_{p-1}^k = v_p B_p^k, \\ \quad \cdots \\ cr\dot{\theta}_i^1 = v_{i+1}B_{i+1}^1, \\ \quad \cdots \\ \dot{\theta}_i^k = v_{i+1}B_{i+1}^k, \\ \quad \cdots \\ \dot{\theta}_{m-1}^1 = v_m B_m^1, \\ \quad \cdots \\ \dot{\theta}_{m-1}^k = v_m B_m^k, \\ \dot{\theta}_m^1 = v_{\theta_m^1} = v_{m+1}B_{m+1}^1, \\ \quad \cdots \\ \dot{\theta}_m^k = v_{\theta_m^k} = v_{m+1}B_{m+1}^k. \end{cases} \tag{28}$$

It is a controlled system on $\mathbb{R}^{k+1} \times (\mathbb{S}^k)^{h+1}$ $(h = m - p + 1)$

$$\dot{\hat{q}} = u_0 \hat{X}_h^0 + \sum_{i=1}^{k} u_i X_h^i,$$

with controls $u_0 = v_m$ and $u_i = v_{\theta_m^i}$, for $i = 1, \cdots, k$.

$$\hat{q} = \Pi_{p,m}(q),$$

$$\hat{X}_h^0 = \sum_{i=p}^{m} f_m^i Z_i + f_m^{p-1}\hat{Z}_{p-1} \ \ et\ \hat{Z}_{p-1} = \sum_{l=1}^{k} \phi_{p-1}^l \frac{\partial}{\partial x_{p-1}^l}.$$

We denote by $\hat{\Delta}_h$ the distribution generated by $\hat{X}_h^0$ and $X_m^1, \cdots, X_m^k$.

This comment will be used in a future paper about singular sets of special flags and their interpretations in terms of singularities kinematic evolution of an articulated arm.

References

[1] J. Adachi, Global stability of special multi-flags, Israel Journal of Mathematics 179 (2010), 29-56.

[2] M. Fliess, J. Levine, P. Martin and P. Rouchon, On differential flat nonlinear systems, Proceedings of the IFAC Nonlinear Control Systems Design Symposium, Bordeaux, France, (1992), 408-412.

[3] F. Jean, The car with n trailers: Characterization of singular configurations, ESAIM: Control, Optimisation and Calculus of Variations 1 (1996), 241-266.

[4] Z.-P. Jiang and H. Nijmeijer, A recursive technique for tracking control of non-holonomic systems in chained form, IEEE Transaction on Automatic Control 44(2) (1999), 265-279.

[5] A. Kumpera and C. Ruiz, Sur l'équivalence locale des systèmes de Pfaff en drapeau, Monge-Ampère Equations and Related Topics, Alta Math. F. Sevi, Rome, (1982), 201-248.

[6] A. Kumpera, Flag systems and ordinary differential equations, Ann. Math. Pura ed Appl. 177 (1999), 315-329.

[7] A. Kumpera and J.-L. Rubin, Multi-flag systems and ordinary differential equations, Nogoya Math. J. 166 (2002), 1-27.

[8] J.-P. Laumond, Controllability of a multibody mobile robot, Proceedings of the International Conference on Advanced Robotics and Automation, Pisa, Italy, (1991), 1033-1038.

[9] J.-P. Laumond, Robot Motion Planning and Control, Lecture Notes on Control and Information Sciences, Springer-Verlag, Berlin, 1997.

[10] S. Li, Géométrie et Classification des Systèmes de Contact: Applications au Contrôle des Systèmes Mécaniques Non Holonomes, PhD Thesis, Rouen, (Février 2010).

[11] R. Montgomery and M. Zhitomirskii, Geometric approach to Goursat flags, Ann. Inst. H. Poincaré -AN 18 (2001), 459-493.

[12] R. Montgomery and M. Zhitomirskii, Points and Curves in the Monster Tower, Memoirs of the AMS, 2009,

[13] P. Mormul, Geometric singularity classes for special k-flags, $k \geq 2$, of arbitrary length, Preprint in: Singularity Theory Seminar, S. Janeczko (ed.), Warsaw Univ. of Technology 8 (2003), 87-100.

[14] P. Mormul, Multi-dimensional Cartan prolongation and special k-flags, H. Hironaka et al. (eds.), Geometric Singularity Theory, Banach Center Publications 65 of Math., Polish Acad. Sci., Warsaw, (2004), 157-178.

[15] P. Mormul, Special 2-flags, singularity classes and associate polynomial normal forms, Contemporary Mathematics and its Applications 33 (2005), 131-145, (in Russian).

[16] P. Mormul and F. Pelletier, Special 2-flags in lengths not exceeding four: A study in strong nilpotency of distributions, Preprint, Warsaw, 2006.

[17] R.-M. Murray, Trajectory generation for a towed cable flight control system, Proc. IFAC World Congress, San Francisco, (1996), 395-400.

[18] W. Pasillas-Lépine and W. Respondek, Applications of the geometry of Goursat structures to non-holonomic control systems, Proceedings of the IFAC Nonlinear Control Systems Design Symposium, Enschede (The Netherlands), (1998), 789-794.

[19] W. Pasillas-Lépine and W. Respondek, On the geometry of Goursat structure, ESAIM: Control, Optimisation and Calculus of Variations 6 (2001), 119-181.

[20] W. Pasillas-Lépine and W. Respondek, Contact systems and corank one involutive sub-distributions, Acta Appl. Math. 69 (2001), 105-128.

[21] W. Pasillas-Lépine and W. Respondek, Nilpotentization of the kinematics of the n-trailer system at singular points and motion planning through the singular locus, International Journal of Control 74 (2001), 628-637.

[22] P. Rodriguez, L'algorithme du Charmeur de Serpents, PhD Thesis, Genève, 2006.

[23] P. Rouchon, Motion planning, equivalence, infinite dimensional systems, Int. J. Appl. Math. Comput. Sci. 11(1) (2001).

[24] K. Shibuya and K. Yamaguchi, Drapeau theorem for differential systems, Differential Geometry and its Application 27(6) (2009), 793-808.

[25] M. Slayman, Bras Articulé et Distributions Multi-drapeaux, Université de Savoie, Laboratoire de Mathématiques (LAMA), PhD Thesis, 2008.

[26] O. J. Sordalen, Conversion of the kinematics of a car with n trailers into a chain form, IEEE Transactions on Automatic Control on Robotics and Automation (1993).

[27] K. Yamaguchi, Contact geometry of higher order, Japanese J. Math. 8 (1982), 109-176.

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